III Algebraic Identities and manipulations

NUMBER THEORY

IIII Permutation and Combinations

I. Divisibility

x divides $y \to x/y \to y = cx \ (c \in I)$

$\uparrow$ factor $\quad \uparrow$ multiple

IMP POINTS:

- $x/y \Rightarrow x/yz \ \forall z \in I$
- x/y and $y/z \Rightarrow x/z$
- x/y and $x/z \Rightarrow x / \alpha y + \beta z \ \forall \alpha, \beta \in I$
- a/b and $b/a \Rightarrow |a| = |b|$
- a/b and $a, b > 0 \Rightarrow |a| \le |b|$
- $a/b \Leftrightarrow ma/mb \ \forall m \in I - \{0\}$
- a/b and $a/c \Rightarrow a^2/bc$
- $a/b \Rightarrow a^k / b^k$
- $x/0 \ \forall x \in I$
- $0/0$; $0 \nmid m \ \forall m \in I - \{0\}$
- x/y and $y \neq 0$ then $\frac{y}{x} / y$
- $x+y \mid x^n + y^n$ if n is odd integer.
- $x-y \mid x^n - y^n \ \forall n \in N$
- $x^2 - y^2 \mid x^n - y^n \ \forall n \in$ even integer$^+$
- Product of r consecutive natural no. will be divisible by $r!$
- $\alpha, \beta, \gamma \in I$ and $n \in N$ then remainder of $\left(\frac{(\alpha\beta + \gamma)^n}{\alpha}\right)$ = remainder of $\left(\frac{\gamma^n}{\alpha}\right)$
- $p \in$ prime and $p / a^2 \Rightarrow p/a, a > 0$

FUNDAMENTAL THEOREM OF ARITHMETIC

Every composite no. can be expressed as a product of primes (factorised) as called prime factorisation which is unique except for order of factors $\to$

DIVISIBILITY TESTS

$2^n \to$ last $n (\in I)$ digits divisible by 2^n

$3^n \to$ sum of digits divisible by 3^n

$5 \to$ unit digit 0 or 5

$7 \to$ sum of 5 time unit digit and remaining no. OR difference of 2 times unit digit and remaining no. divisible by 7

$11 \to$ different b/w sum of digits at even and odd place either 0 or multiple of 11.

composite no. $\to$ divisible by all their factors
$6 \to 3 \times 2$
$12 \to 4 \times 3$ - - -

II. Euclid Division Algorithm / Lemma

for any 2 integers a and b $(a \neq 0)$ there exist a unique pair of integers q and r such that

$$b = qa + r \quad (0 \le r < |a|)$$

quotient (q), remainder (r); $a-1$ remainder, PigeonHole

$\to b = qn - r \equiv qn - r + q \quad (7n - 5 \longleftrightarrow 7n + 2)$

III. Congruent Modulo

Two integers a and b are congruent modulo m i.e, $a \equiv b \pmod m$ if a and b both have same remainder when divided by integer m.

IMP POINTS:

- $a \equiv b \pmod m$
  - $\to a = mk_1 + r$
  - $\to b = mk_2 + r$

  $\to a - b = mk \longleftrightarrow$ $\to m/(a-b)$, $\to a - b \equiv 0 \pmod m$

  if $b < m \to$ a gives b as remainder when divided by m

- $-$ve remainder $(-r) \longleftrightarrow$ +ve remainder $(-r+m)$
- try to make $a \equiv \pm 1 \pmod m$ (finding remainders)
- sum of digits of $n \equiv n \pmod 9 \equiv n \pmod 3$
- in a equation LHS (mod m) = RHS (mod n) (verify, find wrong cases)

PROPERTIES

- $a \equiv b \pmod m \Rightarrow ac \equiv bc \pmod m$; $a^n \equiv b^n \pmod m$ (cyclicity); $a \pm c \equiv b \pm c \pmod m$; $a + k_1 m \equiv b + k_2 m \pmod m$
- $a \equiv b \pmod m$ and $b \equiv c \pmod m \Rightarrow a \equiv c \pmod m$
- $a \equiv b \pmod m$ and $c \equiv d \pmod m \Rightarrow a \pm c \equiv b \pm d \pmod m$, $ac \equiv bd \pmod m$, $ax + by \equiv bx + dy \pmod m$
- $ac \equiv bc \pmod m$ and $(c, m) = 1$ (coprimes) then, $a \equiv b \pmod m$, viceversa.
- $n \equiv 0, 1, 2, 3 \ldots (a-1) \pmod a$

IV. Factors (divisors) and Multiples

n/m

IMP POINTS

- 1 (smallest) and n (largest) always divides $n \to$ unproper divisors (trivial)
- Prime no. do not have proper divisors only 1 and p.
- to check prime, checks its divisibility by primes till $\sqrt{N}$
- $N = D_1 \times D_2$ if $D_1 \geq D_2 \Rightarrow D_1 \ge \sqrt{N}$ & $D_2 \le \sqrt{N}$
- only perfect squares have ~~even~~ odd no. of divisors
- $N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \ldots p_n^{a_n} \to$ primes

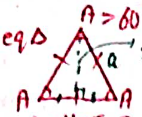
  no. of divisors $= (a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$

  sum of divisors $= (p_1^0 + p_1^1 + p_1^2 + \ldots p_1^{a_1}) \ldots (p_n^0 + p_n^1 + \ldots + p_n^{a_n})$

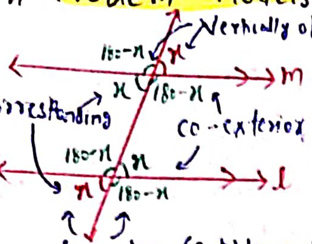
GEOMETRY

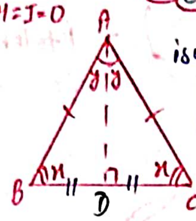
I. Angle Chasing and Side and Area
Problem Models

eq Δ $A=60°$ $\frac{\sqrt{3}}{4}a^2$



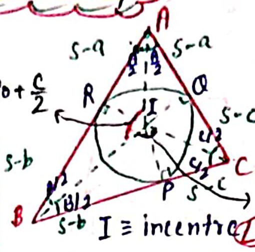
$G=H=I=O$





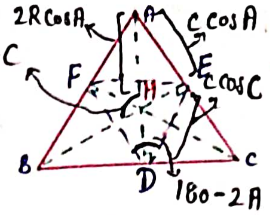
isosceles Δ
$B=C$
$AD \to$ median, Angle bisector, perpendular

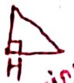
Centres of Triangles



$I \equiv$ incentre (2b)

$IQ=IP=IR=r=\frac{\Delta}{s}$

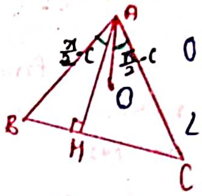




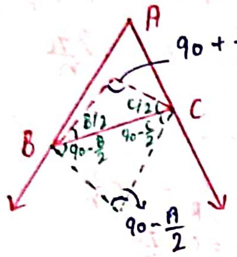
$H \equiv$ orthocentre (4) — in/exc of FED

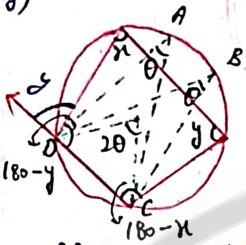
Cyclic Quad $\to$ BDHF, DHEC, AFHE, BCEF, CDFA, BDEA

$\to$ mirror image of H in side lie on circumcircle of ΔABC

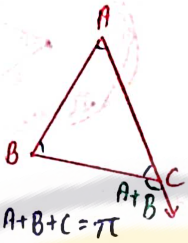


$O \equiv$ circumcentre
$AH \perp BC$
$\angle BAH = \angle CAO$
$= \frac{\pi}{2} - C$

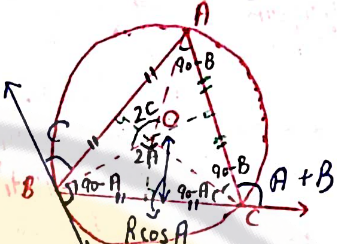




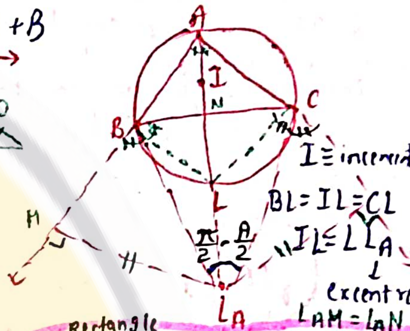
ABCD $\to$ concyclic
vice versa



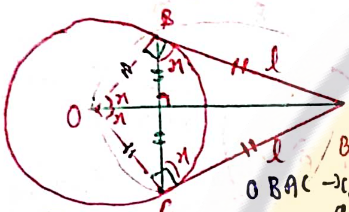
$A+B+C=\pi$



$OA=OB=OC=R=\frac{abc}{4\Delta}$
$\hookrightarrow$ circumcentre (1b)

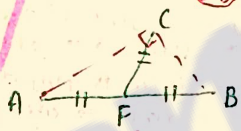


$I \equiv$ incentre
$BL=IL=CL$
$IL \le LI_A$
I_A excentre
$L_AM = L_AN$



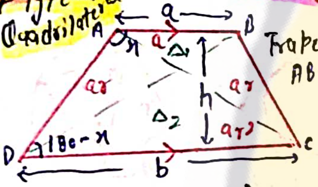
$BC = \frac{2rl}{\sqrt{r^2+l^2}}$

OBAC $\to$ cyclic quad. (kite)
OA $\to$ diameter



$\angle ACB = 90°$
F centre of Δ ACB's circumcircle

Type of Quadrilateral



Trapezium $AB \parallel CD$

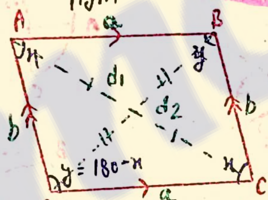
$A = \frac{1}{2}h(a+b)$ $\frac{\Delta_1}{(a)} \sim \frac{\Delta_2}{ar^2}$

isosceles trapezium $\leftrightarrow$ concyclic
vice versa true

Kite

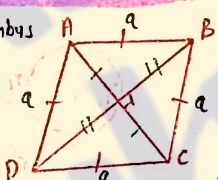
$A = \frac{1}{2}d_1 d_2$

llgm



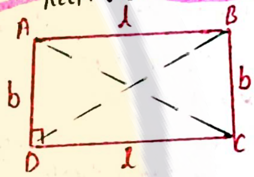
AB = CD, AD = BC
AC $\ne$ BD (bisects too)
A = base × altitude
$2(a^2+b^2) = d_1^2 + d_2^2$

Rhombus



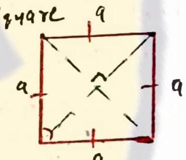
AC $\ne$ BD (bisect $\perp$)
AB = BC = AD = CD
$A = \frac{1}{2}d_1 d_2$

Rectangle



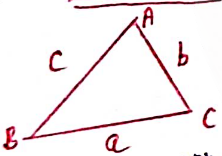
AB = CD
AD = BC
AC = BD
$A = l \times b$
$P = 2(l+b)$

Square



$AC = BD = \sqrt{2}a$ (bisect $\perp$)
$A = a^2$
$P = 4a$

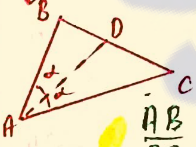
II. Triangles



$\to$ side length $\propto$ opposite angle
$\to$ sum of any two sides > third side
$\to$ diff. of any 2 sides < third side
$\to A+B+C=\pi$
$\to \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (circumradius)
$\to \cos A = \frac{b^2+c^2-a^2}{2bc}$
$\to A = \frac{1}{2}b\cdot h = \frac{1}{2}ab\sin C = \Delta$
$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$
$\to$ congruency criterion ($A_i=B_i, a_i=b_i$)
SAS, ASA, SSS, RHS, AAS
$\to$ Similarity criterion ($A_i=B_i, \frac{a_i}{b_i}=\frac{b_i}{b_i}=\frac{c_i}{c_i}$)
AAA, AA, SAS, SSS $\left(\frac{\Delta_1}{\Delta_2} = \left(\frac{a}{b}\right)^2\right)$
$\to$ ratio of area of similar Δ's equal square of ratio of their sides, median, altitude, angle bisector, inradius, circumradius
$\to$ triangles b/w same parallel and equal bases have equal area and vice versa

Angle Bisector

Each point on angle bisector is equidistant from sides



Divides opp. side internally in ratio of containing sides

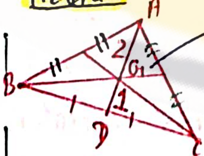
$AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

$\frac{BD}{DC} = \frac{c}{b}$

$\frac{AB}{BD} = \frac{AC}{CD}$

$BD = \frac{c}{c+b}\cdot a$ $CD = \frac{b}{b+c}\cdot a$

Median

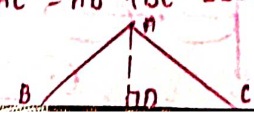


Divides opp side in half
$\to$ centroid
$\hookrightarrow$ divides median 2:1
divides H & O in 2:1 H G O
All six Δ area equal

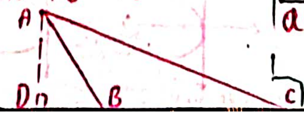
Sum of median < sum of sides
2 AD < AB + AC
$AD = \frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$ (Appollonius)
$M_a^2 + M_b^2 + M_c^2 = \frac{3}{4}(a^2+b^2+c^2)$

Altitude ($\perp$ from vertex)

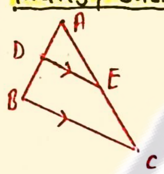
Acute Angle Theorem
$AC^2 = AB^2 + BC^2 - 2BD \times BC$



Obtuse Angle Theorem
$AC^2 = AB^2 + BC^2 + 2BD \times BC$



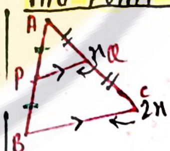
Thales / Basic Proportionality



$DE \parallel BC \leftrightarrow \frac{AD}{DB} = \frac{AE}{CE}$

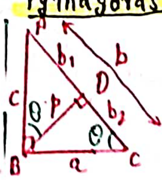
$\frac{AB}{AD} = \frac{AC}{AE} = \frac{DE}{BC}$

Mid Point Theorem



P, Q $\to$ mid points
$PQ \parallel BC$
and vice versa
Also, $PQ = \frac{1}{2}BC$

Pythagoras Theorem



$\Delta ABD \sim \Delta BCD \sim \Delta ABC$
$c^2 = b_1 b$ $p^2 = b_1 b_2$
$a^2 = b_2 b$ $a^2+c^2=b^2$
$ac = pb$

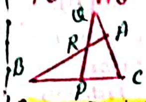
D $\to$ mid point $\to$ circumcentre $\to p = b_1 = b_2$

Ceva's Theorem

$\frac{BX}{CX}\cdot\frac{CY}{AY}\cdot\frac{AZ}{ZB} = 1$

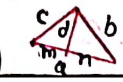
Menelau's Theorem

$\frac{BP}{PC} \times \frac{CQ}{AQ} \times \frac{AR}{RB} = 1$



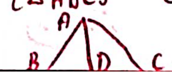
Stewart's Theorem

$a(d^2+mn) = b^2m + c^2n$



Base Division Theorem

$\frac{[\Delta ABD]}{[\Delta ADC]} = \frac{BD}{CD}$



Trigonometry Identities

Identities (squaring, adding eq")

- $\sin^2\theta + \cos^2\theta = 1$
- $\sec^2\theta - \tan^2\theta = 1$
- $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Compound angled

- $\sin(A+B) = \sin A\cos B + \cos A\sin B$
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

Sum to Product

- $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
- $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Product to Sum

- $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
- $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
- $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
- $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

- $\tan\left(\frac{\pi}{4}+\theta\right) = \dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} = \dfrac{1+\tan\theta}{1-\tan\theta}$
- $\tan\left(\frac{\pi}{4}-\theta\right) = \dfrac{\cos 2\theta}{1+\sin 2\theta} = \dfrac{1-\tan\theta}{1+\tan\theta}$
- $\dfrac{1+\sin x}{\cos x} = \sec x + \tan x = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

Double Angle

- $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$
- $\cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$
- $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \tan((\theta+A)+(\theta-A))$, $\underset{\alpha}{\uparrow} + \underset{\beta}{\uparrow}$

Triple Angle

$\dfrac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta$

- $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
- $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ — $\dfrac{\cos 3\theta}{\cos\theta} = 4\cos^2\theta - 3$
- $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

Vn method

Half Angle

- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
- $\tan\left(\frac{\theta}{2}\right) = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

Miscellaneous

- $\sin 2x = 2\sin x\cos x = (\sin x + \cos x)^2 - 1$
  $= 1 - (\sin x - \cos x)^2$
- $1 + \sin\theta = \left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2$
- $\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \frac{1}{2}\sin^2 2\theta$
- $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta = 1 - \frac{3}{4}\sin^2 2\theta$
- ($3x = x + 2x$, apply tan) $\tan x \times \tan 2x \times \tan 3x = \tan 3x - \tan 2x - \tan x$
- $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta}$ $\quad$ $\cot\theta = \dfrac{1+\cos 2\theta}{\sin 2\theta}$
- Vn method: $\tan A + \tan B = \dfrac{\sin(A+B)}{\cos A\cos B}$ $\quad$ $\tan A - \tan B = \dfrac{\sin(A-B)}{\cos A\cos B}$
- $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$
- $\cot\theta - \tan\theta = 2\cot 2\theta$
  → $\tan(\ ) \to$ telescopic series

Conditional Identities

- $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$
- $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = -\sin^2 A + \cos^2 B$
- $\sin\theta\,\sin(60^\circ+\theta)\sin(60^\circ-\theta) = \frac{1}{4}\sin 3\theta$
- $\cos\theta\,\cos(60^\circ+\theta)\cos(60^\circ-\theta) = \frac{1}{4}\cos 3\theta$
- $\tan\theta\,\tan(60^\circ+\theta)\tan(60^\circ-\theta) = \tan 3\theta$
- cos Angle in GP $r=2$: $\cos\alpha\,.\cos 2\alpha\,.\cos 2^2\alpha\,.\cos 2^3\alpha\ldots\cos(2^{n-1}\alpha) = \dfrac{\sin(2^n\alpha)}{2^n\sin\alpha}$ — $\dfrac{2\sin\alpha}{2\sin\alpha}$
- cos Angle in AP: $\cos A + \cos(A+B) + \cos(A+2B)\ldots + \cos(A+(n-1)B)$
  $= \cos\left(A + \frac{(n-1)B}{2}\right)\left(\dfrac{\sin\left(\frac{nB}{2}\right)}{\sin\left(\frac{B}{2}\right)}\right)$ — $\dfrac{2\sin\alpha}{2\sin\alpha}$, $\alpha = B/2$
- sin Angle in AP: $\sin A + \sin(A+B) + \sin(A+2B)\ldots + \sin(A+(n-1)B)$
  $= \sin\left(A + \frac{(n-1)B}{2}\right)\left(\dfrac{\sin\left(\frac{nB}{2}\right)}{\sin\left(\frac{B}{2}\right)}\right)$ — $\dfrac{2\sin\alpha}{2\sin\alpha}$, $\alpha = B/2$

$\boxed{\text{If } A+B = 90^\circ}$

- → $\sin^2 A + \sin^2 B = 1$ — $\sin^2 A$ adding series
- → $\sin^4 A + \sin^4 B = \dfrac{1-\sin^2 2A}{2} = \dfrac{1-\sin^2 2B}{2}$
- sin multiply series → $\sin A\sin B = \dfrac{\sin 2A}{2} = \dfrac{\sin 2B}{2}$

$\boxed{\text{If } A+B+C = \pi}$

- → $\sin(A+B) = \sin C$ $\quad$ → $\cos(A+B) = -\cos C$
- → $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$
- → $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$
- → $\cos A + \cos B + \cos C = 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$
- → $\sum\tan A = \prod\tan A$ $(A+B+C = n\pi)$
- → $\sum\tan\frac{A}{2}\tan\frac{B}{2} = 1$
- → $\sum\cot\frac{A}{2} = \prod\cot\frac{A}{2}$
- → $\sum\cot A\cot B = 1$

Application of Derivative

I. Rate Of Change

$f'(x) = \frac{dy}{dx}$ = rate of change of y w.r.t. x

- $y' > 0$ (y increasing) $y' < 0$ (y decreasing) $y' = 0$ (y constant)
- $\frac{dy}{dx} = k \rightarrow dy = k\,dx$ (Integrate to get expression of y) (constant rate of change)
- Mensuration, Geometry formulaes, make relation in one variable, diff^n both side, use relations to get required result
- Length $\underset{y'}{\overset{\int}{\rightleftarrows}}$ Area $\underset{y'}{\overset{\int}{\rightleftarrows}}$ Volume

Dist. $\underset{y'}{\overset{\int}{\rightleftarrows}}$ Speed $\underset{y'}{\overset{\int}{\rightleftarrows}}$ acceleration $\underset{y'}{\overset{\int}{\rightleftarrows}}$ jerk

II. Approximations & Errors (Δx)

- $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \xrightarrow{\sim} \Delta y = \left(\frac{dy}{dx}\right)\Delta x$
- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\hookrightarrow f(x+h) = f(x) + h\,f'(x)$

- Used to find $(126)^{1/3} \Rightarrow y = x^{1/3}$, $x = 125$, $\Delta x = 1$
- Used to find $f(1.01) \Rightarrow f(1+0.1)$

⊕ Rolle's & LMVT & Curve Tracing
↳ Apply again & again in intervals getting some inequality / equality, find ↳ min/max value

III Tangents and Normal

→ curve, $y = f(x)$, slope/gradient at $P(x_1, y_1)$ ie point of tangency is

$m_t = \tan\theta = \left.\frac{dy}{dx}\right|_{(x_1,y_1)}$ ($\theta \to$ angle b/w tangent and +ve x axis)

$= \frac{y_2 - y_1}{x_2 - x_1}$

↳ Normal at P is ⊥ to tangent

$m_t = -\frac{1}{m_n}$

- Eq^n tangent: $y - y_1 = m_t(x - x_1)$
- Eq^n normal; $y - y_1 = m_n(x - x_1) = -\frac{(x - x_1)}{m_t}$

→ Point of tangency, P satisfies tangent, normal and curve

→ slope of tangent line = slope of (dy/dx) curve at point of tangent.

Parametric Coordinates

e.g. circle, parabola, ellipse,
$y^2 = x^3\ (t^2, t^3)$, $x^{2/3} + y^{2/3} = a^{2/3}\ (a\cos^3\theta, a\sin^3\theta)$

Angle B/W Curves

= Angle b/w tangents as point of intersection

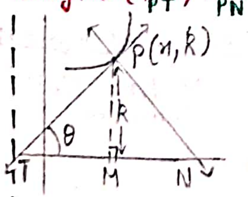
$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ (acute)

→ if C_1, C_2 cuts at 90° then curves are orthogonal at P (assume PoI (α,β) to check)

→ if ∀ x at all point of intersection
$\tan\theta = N.D$, $\theta = 90°$, $m_1 m_2 = -1$
then curves are Orthogonal curves

→ Both curve touch each other, $m_1 = m_2$, $\tan\theta = \theta = 0°$ & intersect at atleast 1 pt

Length (tan, Nor, Sub-tan, Sub-Nor)
PT, PN, TM, MN



$m = \tan\theta = \left.\frac{dy}{dx}\right|_{P(x,k)}$

$PT = k\,\text{cosec}\,\theta = \left|k\sqrt{1 + 1/m^2}\right|$

$PN = k\sec\theta = \left|k\sqrt{1+m^2}\right|$

$TM = k\cot\theta = \left|k/m\right|$

$MN = k\tan\theta = |km|$

length of curve f(x) from x=a to x=b
$= \int_a^b \sqrt{1 + (f'(x))^2}\,dx$

* Imp. Points

- tangent cuts coordinate axes at equal length OR equally inclined at coordinate axes
$\Rightarrow m = \pm 1$
- tangent make equal non-zero intercept at coordinate axes
$\Rightarrow m = -1$
- $\frac{dy}{dx} = 0$ tangent ∥ x axis
- $\frac{dx}{dy} = 0$ or $\frac{dy}{dx} \to \infty$ tangent ∥ y axis
- ⊥ lines (tangent) $m_1 m_2 = -1$ (orthogonal)
- ∥ lines (tangent) $m_1 = m_2$ (curve touch each other)
- Bitangent: tangent to curve at 2 distinct points
- tangent may intersect to curve at another point(s) on curve
- tangent can cross curve at point of tangency (x^3 ($x=0$))
- tangent can exist at P even if $\frac{dy}{dx}$ do not exist at P ($x^{2/3}$ ($x=0$))

Shortest Distance b/w curves/ lines / points

↳ shortest distance for non-intersecting curves is always along common normal (∥ tangent)

→ find general point (assume), Equate slope of normal/tangent, hence find required

Monotonicity of Composite f^n

$h(x) = g(f(x)) \rightarrow h'(x) = g'(f(x))\,f'(x)$

↳ critical points, $g'(f(x)) = 0$ OR $f'(x) = 0$

$f(x), g(x)$ both ↑ OR ↓ → $h(x)$ ↑

$f(x), g(x)$ one ↑ other ↓ → $h(x)$ ↓

IV Increasing f^n and Decreasing f^n

for $x_2 > x_1$ ⇒ $f(x_2) \geq f(x_1)$ (inequality) | for $x_2 > x_1$ ⇒ $f(x_2) \leq f(x_1)$ (inequality)

i.e, $f(a-h) < f(a) < f(a+h)$ | i.e, $f(a-h) > f(a) > f(a+h)$
∀ x ∈ Interval

* f(a) should be definite, no need to check cont. & diff^n

Increasing f^n → $\frac{dy}{dx} \geq 0$ | Decreasing f^n → $\frac{dy}{dx} \leq 0$

* for strictly ↑/↓ → remove equality
* $\frac{dy}{dx} = 0 \rightarrow x = a$ f^n can be inc. or dec. provided $f'(x)$ doesn't change sign in neighbourhood of $x = a$

i.e, Equality holds for discrete values of x,
if intervals satisfy then equality doesn't hold.

* Check where f^n definition changes (piecewise f^n)

→ Monotonic f^n: either inc. or dec. ∀ x ∈ D
→ Non-monotonic f^n: inc. & dec. e.g. constant f^n, |x|, sinx

* To prove $f(x) \geq g(x)$ ∀ x ∈ [a,b] assume, $h(x) = f(x) - g(x)$ ie to prove $h(x) \geq 0$ ∀ x ∈ [a,b]

i.e, $a \leq x < b \rightarrow f(a) - g(a) \leq f(x) - g(x) < f(b) - g(b)$
(inequality changes acc. to inc/dec f^n)

Use monotonicity

→ Critical Points

$x = c$ is critical point of f(x) if f(c) exist and
$f'(c) = \left.\frac{dy}{dx}\right|_c = 0$ OR $f'(c) = N.D$ i.e, $\left.\frac{dx}{dy}\right|_c = 0$

* end points of interval are not included in critical points

→ Stationary Points
$f'(c) = 0$ and f(c) exist (defined)

Sign of f'(x)

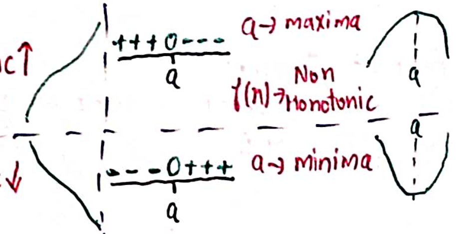
+++0+++ (a) f(x) → inc↑ at x = a

---0--- (a) f(x) → dec↓ at x = a

+++0--- (a) a → maxima

---0+++ (a) a → minima

Non f(x) → Monotonic



No. of functions

n → m, $m \geq n$
no. of ↑↑/↓↓ f^n = ^mC_n

no. of ↑↑/↓↓ f^n = $^{m+n-1}C_n$ = $^{m+n-1}C_{m-1}$

Graph

→ linear equations, linear inequalities, wavy curve (method), Domain, Range, Codomain

shows relationship b/w two quantities

x	a_1	a_2	...
y	b_1	b_2	...

All Domain values

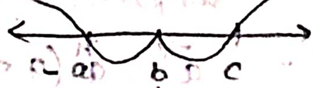
comes All Range values

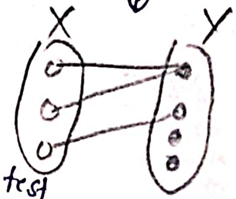
Plot

(o, •, shading)

family of curves

Linear Programming





Vertical line draw, A.O.D. → ↑, ↓, max, min,

to draw more than 1 graph in one cartesian plan

$\tan\theta / \frac{dy}{dx}$, 1st, 2nd derivative test, Tangent, Normal

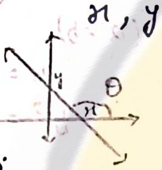
Polynomial function

① Straight lines

slope (+ve/−ve/0) — intercept x, y

$\tan\theta = m$

$m = \frac{y_2 - y_1}{x_2 - x_1}$



$m \begin{cases} +ve \to \theta < 90° \\ 0 \to \theta = 90° \\ -ve \to \theta > 90° \end{cases}$

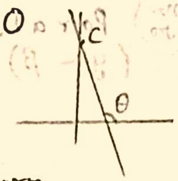
x	0	a
y	b	0

$\perp \Rightarrow m_1 m_2 = -1$

(i) General form

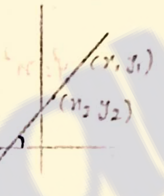
$Ax + By + C = 0$

$m = -\frac{A}{B}$



(ii) Slope point form

$y - y_1 = m(x - x_1)$



(iii) Two point form

$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

(iv) Slope intercept form

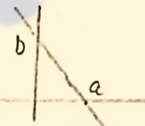
$y = mx + c$

y intercept

(v) Double intercept

$\frac{x}{a} + \frac{y}{b} = 1$

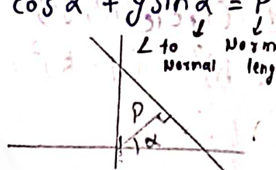
x intercept, y intercept



(vi) Normal form

$x\cos\alpha + y\sin\alpha = p$

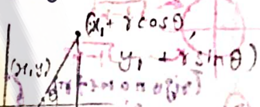
∠ to Normal, Normal length



(vii) Parametric form

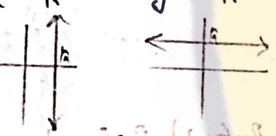
$\frac{y - y_1}{\sin\theta} = \frac{x - x_1}{\cos\theta} = r$

$(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$



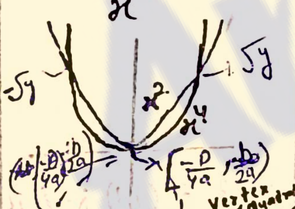
② ⇒ constant function

$x = k$ $y = k$

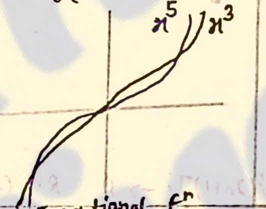


Polynomial f^n

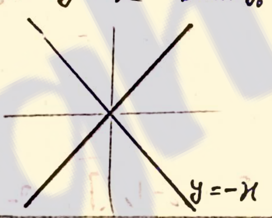
x^{2n}



x^{2n+1}



$y = x$ Identity f^n



\# One-One: 1 x ⟷ 1 y

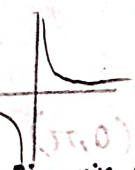
Many one: more than 1 x ⟷ 1 y

Onto: range = codomain

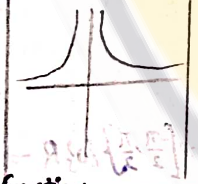
Into: range ≠ codomain

③ Rational f^n

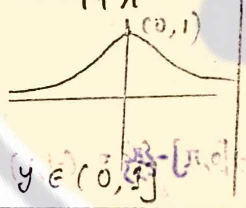
$y = \frac{1}{x}$, $x \neq 0$



$y = \frac{1}{x^2}$, $x \neq 0$



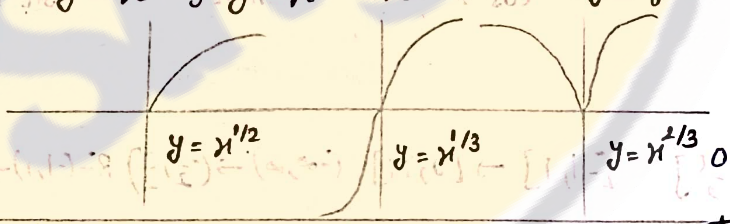
$y = \frac{1}{1+x^2}$



$y \in (0, 1]$

④ Irrational f^n

$y = x^n$ & $y = x^{1/n}$ are mirror image of each w.r.t. $y = x$



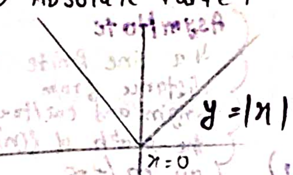
* $y = x^n$

$x > 1 \to x < x^2 < x^3$

$0 < x < 1 \to x > x^2 > x^3$

⑤ Piecewise functions

Absolute value f^n



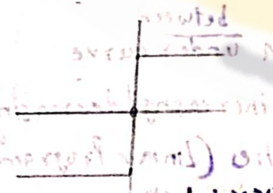
$f(x) = |x| \begin{cases} x ; x > 0 \\ -x ; x < 0 \\ 0 ; x = 0 \end{cases}$

$x = 0$ critical points

→ $y = |ax+b|$, $-b/a$

for multiple modulas find critical points and examine the behavior of expression in each interval (same)

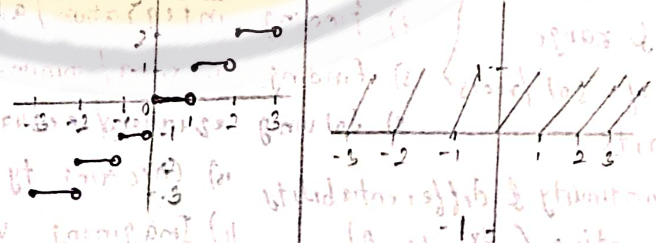
Signum f^n



$f(x) = \frac{|x|}{x} \begin{cases} 1 ; x > 0 \\ -1 ; x < 0 \\ 0 ; x = 0 \end{cases}$

GIF $y = [x]$

FPF $y = \{x\}$



Discontinuous at integral values

max/min $\{f(x), g(x)\}$

Plot $f(x)$ and $g(x)$

required graph = max/min of $f(x)$ or $g(x)$ $\forall x$

\# Shifting of Origin to (h, k) without rotation

no change in curve

$(x, y) \to (x - h, y - k)$

$f(x, y) \to f(x + h, y + k)$

\# Even f^n

$f(-x) = f(x)$

"symmetrical about y axis"

\# Odd f^n

$f(-x) = -f(x)$

"symmetrical about origin"

if $f(a+x) = f(a-x)$ then graph is symmetrical about $x = a$